Lecture 32

Applications to Area, Arc length, Volume and Surface area

1 Area

Suppose $f(x) \geq 0$ on $[a, b]$. Then it is clear from the definition of Definite integral that the area under the curve $y = f(x)$ can be approximated by Riemann sums. i.e.,

$$A \cong \sum_{k=1}^{n} f(\xi_i)(x_i - x_{i-1}) \to \int_a^b f(x)dx \quad \text{as } n \to \infty.$$

Similarly, the area bounded by the curves $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$ on $[a, b]$ is

$$A = \int_a^b (f(x) - g(x))dx.$$

Problem 1.0.1. *Find the area bounded by $y = x^2$ and $y^2 = x$*

Solution: The curves interest at $x = 0, 1$. The upper curve is $y^2 = x$ and lower curve is $y^2 = x$. So by the above formula

$$A = \int_0^1 (\sqrt{x} - x^2)dx = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

One can also find by integrating along y: $A = \int_0^1 (\sqrt{y} - y^2)dy = \frac{1}{3}$.

Polar coordinates

A point (x, y) on the xy-plane is assigned polar coordinates (r, θ) if the point is at a distance $r = \sqrt{x^2 + y^2}$ from the origin on the ray at an angle θ with positive x-axis. We allow r negative with convention: $(-r, \theta) = (r, \theta + \pi)$. Each point on the plane has infinitely many representations in polar form, for example $(1, 0)$ is at a distance of 1 units from the origin on the x-axis. So it can be represented in polar form also as $(r, \theta) = (1, 0)$. Also it is same as $(1, 2n\pi), n \in \mathbb{N}$ and $(-1, \pi)$. Each point (r, θ) is same as $(r, \theta + 2n\pi)$ for all $n \in \mathbb{N}$.

Example 1.0.1. *The point $(2, \pi/6)$ can also be represented by $(-2, \frac{5\pi}{6})$ and $(-2, \frac{7\pi}{6})$*

Relation with cartesian coordinates:

We often use the following relations:

1. Given the polar coordinates $(r\theta)$, we can write the cartesian coordinates using $x = r\cos\theta$, and $y = r\sin\theta$.

2. Given the cartesian coordinates (x, y), we can write polar coordinates using $r = \sqrt{x^2 + y^2}$, and $\theta = \tan^{-1}(\frac{y}{x})$

Circles and Straight lines:

1. The circle $x^2 + y^2 = a^2$ in cartesian coordinates, using (1) above, $r^2 \cos^2\theta + r^2 \sin^2\theta = a^2$ which is $r = a$.

2. The circle $(x-a)^2 + y^2 = a^2 \implies x^2 + y^2 - 2ax = 0$, again by (1) above we get $r^2 - 2ar\cos\theta = 0 \implies r = 2a\cos\theta$.

3. The circle $x^2 + (y-a)^2 = a^2 \implies x^2 + y^2 - 2ay = 0$, again by (1) above we get $r^2 - 2ar\sin\theta = 0 \implies r = 2a\sin\theta$.

4. The straight line $y = mx$ is $\theta = \tan^{-1} m$

5. The straight line $x = a$ is $r = a\sec\theta$ and $y = b$ is $r = b\csc\theta$.

Symmetry in polar coordinates: The symmetry of the graph of the function in polar coordinates helps one to plot/trace the graph. There are three types of symmetry principles.

1. For (r, θ) on the graph, suppose $(r, -\theta)$ is also on the graph. Then the graph is symmetric about $x-$ axis.

2. For (r, θ) on the graph, suppose $(r, \pi - \theta)$ is also on the graph. Then the graph is symmetric about y- axis.

3. For (r, θ) on the graph, suppose $(r, \pi + \theta)$ is also on the graph. Then the graph is symmetric about the origin.

Examples 1.0.2. *1. (leminiscate): Consider the function* $r^2 = \cos 2\theta$. *If* (r, θ) *is on the graph, then* $r^2 = \cos 2(-\theta) = \cos 2\theta$ *implies* $(r, -\theta)$ *is also on the graph. So the graph is symmetric about* $x-$ *axis.*

Again, $r^2 = \cos 2(\pi - \theta) = \cos(2\pi - 2\theta) = \cos 2\theta$ *implies* $(r, \pi + \theta)$ *is also on the graph. Therefore, graph is symmetric about y-axis.*

We can also see that $(r, \pi + \theta)$ *is also on the graph. So the graph is also symmetric about the origin.*

Hence it is enough to trace the curve in the first quadrant. Now since $r^2 \geq 0$, *the domain of* θ *in the first quadrant is* $[-\frac{\pi}{4}, \frac{\pi}{4}]$. *Also one can see by the derivative test that* $\theta = 0$ *is a point of local maxima(see figure 1).*

2. (Cardioid): Consider the function $r = 1 - \cos\theta$. *Then if* $(r, \theta) \in$ *graph* $\implies (r, -\theta) \in$ *graph. So the graph is symmetric with respect to* $x-$ *axis. So it is enough to trace the curve for* $0 \leq \theta \leq \pi$. *Again by derivative test we see that* $\theta = 0$ *is a point of minimum and* $\theta = \pi$ *is point of maximum(see figure 2).*

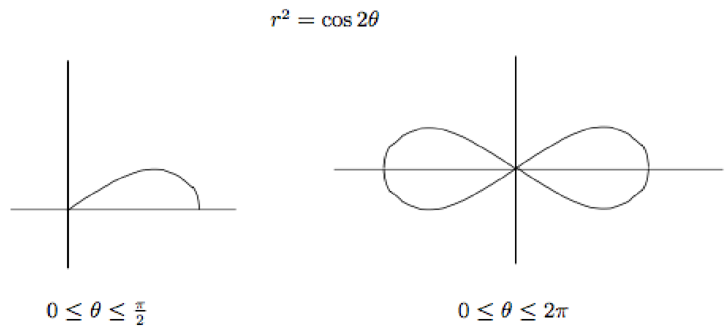


Figure 1: leminiscate

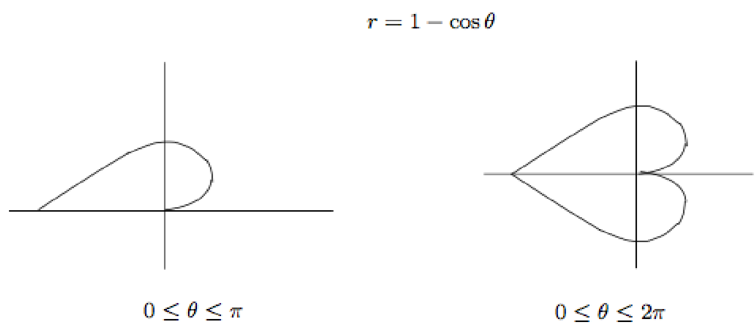


Figure 2: Cardioid

Area in polar coordinates: Let a region be bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$. We approximate the region with n non-overlapping circular sectors based on the partition P of angle $\theta \in [\alpha, \beta]$. The typical sector has radius $r_k = f(\theta_k)$ and central angle of radian measure $\Delta\theta_k$. Its area is $\frac{\Delta\theta_k}{2\pi}$ times the area of a circle r_k. i.e.,

$$A_k = \frac{1}{2} r_k^2 \Delta\theta_k = \frac{1}{2} f(\theta_k)^2 \Delta\theta_k$$

The area of the region is approximately $\sum_{k=1}^{n} A_k$. Taking $n \to \infty$ so that $\|P\| \to 0$, we get

$$A = \int_\alpha^\beta \frac{1}{2} f(\theta)^2 d\theta = \frac{1}{2} \int_\alpha^\beta r^2 d\theta.$$

Problem 1.0.2. *Find the area of the region enclosed by the cardioid* $r = 2(1 - \cos\theta)$.

Solution: From the graph discussed above, the range of θ is from 0 to 2π. Therefore, the area is

$$A = \frac{1}{2}\int_0^{2\pi} r^2 d\theta = \int_0^{2\pi} (3 + \cos 2\theta - 4\cos\theta) d\theta = 6\pi.$$

2 Arc length

Consider a curve defined by $y = f(x)$ between $x = a$ and $x = b$. For example $y = \sin x$ between $x = 0$ and π. The length of this curve can be approximated by sum of lengths of straight lines connecting $(0, 0) \to (\pi/4, \sin(\pi/4)) \to (\pi/2, \sin(\pi/2)) \to (\pi, 0)$. The arc length s is approximately

$$\sqrt{(\frac{\pi}{4})^2 + (\frac{1}{\sqrt{2}})^2} + \sqrt{\frac{\pi}{4})^2 + (1 - \frac{1}{\sqrt{2}})^2} + \sqrt{(\frac{\pi}{2})^2 + 1}.$$

This approximation becomes better and better as we refine the partition $\mathcal{P} = \{0, \pi/4, \pi/2, \pi\}$. For a given curve defined by function $y = f(x)$ between $x = a, b$, we consider the partition $\mathcal{P} = \{a = x_0, x_1, x_2, \ldots.x_{k-1}, x_k, \ldots x_n = b\}$. Then the length of this curve may be approximated by the formula

$$\begin{aligned} s &\sim \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) + f(x_{i-1}))^2} \\ &= \sum_{i=1}^{n} \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2}\ (x_i - x_{i-1}) \\ &\to \int_a^b \sqrt{1 + (f'(x))^2} dx \ \text{ as } \ n \to \infty \end{aligned}$$

The following two formulas are used for finding the *Arc length* or *length of curve*:

1. For a function $y = f(x)$ between $x = a$ and $x = b$

$$s = \int_a^b \sqrt{1 + \left(\frac{df}{dx}\right)^2}\ dx.$$

2. For a function $x = f(y)$ between $y = c$ and $y = d$

$$s = \int_c^d \sqrt{1 + \left(\frac{df}{dy}\right)^2}\ dy.$$

Parametric form: Suppose if an arc is defined in the parametric form $x = x(t), y = y(t)$ between $t = T_1$ and $t = T_2$. Then we note from above approximation, that s may be approximated by taking the partition $\mathcal{P} = \{T_1 = t_0, t_1, ..., t_n = T_2\}$ and

$$
\begin{aligned}
s \sim & \sum_{i=1}^{n} \sqrt{\left(\frac{x_i - x_{i-1}}{t_i - t_{i-1}}\right)^2 + \left(\frac{y_i - y_{i-1}}{t_i - t_{i-1}}\right)^2} (t_i - t_{i-1}) \\
\to & \int_{T_1}^{T_2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \text{ as } n \to \infty.
\end{aligned}
$$

Problem 2.0.1. *Find the arc length of the curve defined by* $x = 2\cos^2\theta,\ y = 2\cos\theta\sin\theta$ *,* $0 \le \theta \le \pi$*.*

Solution: This curve is a circle with radius 1 at $(1, 0)$. So the answer should be 2π. Applying formula

$$
\begin{aligned}
s = & \int_0^{\pi} \sqrt{x'(\theta)^2 + y'(\theta)^2}\, d\theta = 2\int_0^{\pi} \sqrt{(2\cos\theta\sin\theta)^2 + (\cos^2\theta - \sin^2\theta)^2}\, d\theta \\
= & 2\int_0^{\pi} \sqrt{\cos^4\theta + 2\cos^2\theta\sin^2\theta + \sin^4\theta}\, d\theta = 2\pi
\end{aligned}
$$

Problem 2.0.2. *Find the arc length of an arc of* $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$*.*

Solution: Take the parametrization: $x = \cos^3 t, y = \sin^3 t, 0 \le t \le \frac{\pi}{2}$. Then

$$
\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9\cos^2 t \sin^2 t.
$$

Hence the arc length is

$$
l = 3\int_0^{\frac{\pi}{2}} \cos t\ \sin t\ dt = \frac{3}{2}
$$

Problem 2.0.3. *Find the approximate value of the length of ellipse* $x = a\cos t, y = b\sin t, 0 \le t \le 2\pi$ *when* $a = 1, e = 1/2$*.*

Solution: By the arc length formula,

$$
l = 4a\int_0^{\pi/2} \sqrt{1 - e^2\cos^2 t} dt
$$

where e is ellipse's eccentricity. This integral is non-elementary except when $e = 0$ or 1. The integrals in this form are called *elliptic integrals*. We can use Trapezoidal rule to evaluate the value when $a = 1$ and $e = 1/2$. The answer with $n = 10$ is $l = 5.870$.

Suppose the curve is given in polar form $r = f(\theta),\ \alpha \le \theta \le \beta$. Then by taking the parametrization $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$ with $\theta \in [\alpha, \beta]$, we get

$$
\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta,\ \ \frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta.
$$

Hence the arc length is

$$l = \int_{\alpha}^{\beta} \sqrt{f^2(\theta) + (f')^2(\theta)} d\theta.$$

Application to Work done

Suppose the force $f(x)$ depends on position x is along a straight line from $x = a$ to $x = b$. Let $n \in \mathbb{N}$, $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ for $i = 1, 2, ..., n$. Then the work done in moving a particle under the force $f(x)$ from x_{k-1} to x_k is approximately $W_k = f(x_k)\Delta x$. The total work done (in moving from a to b) approximately is $W \sim \sum_{i=1}^{n} f(x_k^*)\Delta x, x_k^* \in [x_{k-1}, x_k]$. Taking $n \to \infty$ we get the total work done as $W = \int_a^b f(x)dx$.

Example: Find the work required to compress a spring from its natural length of 1 foot to a length of 0.75 foot if the force constant is $k = 16$ kg/foot.

Solution: Hooks law ways that the force it takes to stretch or compress a spring x length nits from its natural length is proportional to x. i.e.,$F = kx$, k is constant measured in force units per unit length.

Suppose the given springs is placed on the $x-$ axis. It is fixed at $x = 1$ and movable end at the origin. From the above formula, the force required to compress the spring from 0 to x with the formula $F = 16x$. To compress the spring from 0 to 0.25 ft, the force must increase from $F(0) = 0$ to $F(0.25) = 16 \times 0.25 = 4$ foot-kg. Therefore, the work done by F over this interval is

$$W = \int_0^{0.25} 16x dx = 0.5 \ ft - kg.$$

Lecture 33

3 Volume of symmetrical objects

Method of Slicing:

Consider a solid lying alongside some interval $[a, b]$ of the x-axis. For each x let $A(x)$ be the area of the cross section (of the solid) obtained by cutting it with a plane perpendicular to the x-axis at x. We divide the interval into n subintervals $[x_{i-1}, x_i]$. The planes that are perpendicular to the x-axis at the points $x_0, x_1, x_2, ..., x_n$ divide the solid into n slices. If the cross section between $[x_{i-1}, x_i]$ changes "little bit" along the that subinterval, then it can be approximated by a cylinder of height $x_i - x_{i-1}$ with base $A(x_i^*), x_i^* \in [x_{i-1}, x_i]$. So the volume of the slice is $V_i = A(x_i^*)(x_i - x_{i-1})$. Then the volume of the solid can be approximated as

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} A(x_i^*)(x_i - x_{i-1}) \to \int_a^b A(x)dx$$

as $n \to \infty$. Now this can be done along any axis, say y-axis. In this case we get the formula:

$$V = \int_a^b A(y)dy.$$

Solid of Revolution: Consider the area between the function $y = f(x), x \in [a, b]$ and x-axis. By revolving this area along $x-$ axis, we obtain a solid which is called "solid of revolution". It is easy to see that for this solid, the cross section is disc of radius $f(x)$ and the area of cross section $A(x)$ is equal to $\pi[f(x)]^2$. Hence the volume is

$$V = \int_a^b A(x)dx = \pi \int_a^b f^2(x)dx.$$

For example, take a cone of radius r and height h. Then this cone can be obtained by revolving $y = \frac{rx}{h}$ about $x-$ axis between $x = 0, h$. Then the volume is

$$V = \pi \int_0^h \frac{r^2x^2}{h^2}dx = \frac{\pi r^2 h}{3}.$$

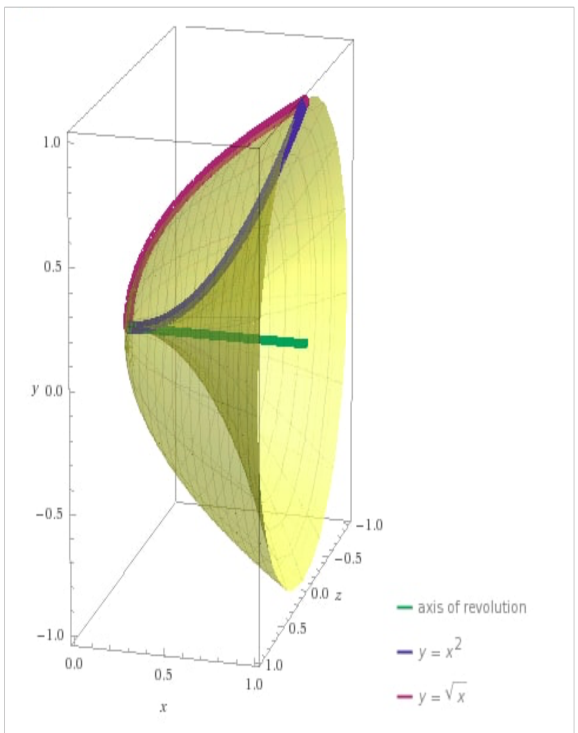
Suppose the area revolved is bounded by two curves $y = f(x) \geq 0$ and $y = g(x) \geq 0$, with $f(x) \geq g(x)$. Then each cross section looks like washer with outer radius $r_1(x) = f(x)$ and inner radius $r_2(x) = g(x)$. The area of the cross section is $\pi(f^2(x) - g^2(x))$. The volume of the solid is

$$V = \pi \int_a^b (r_1^2 - r_2^2)(x)dx = \pi \int_a^b (f^2(x) - g^2(x))dx.$$

If the revolution is performed about y axis. Then

$$V = \pi \int_a^b (f^2(y) - g^2(y))dy$$

Example 3.0.1. *The volume of the solid obtained by revolving the area bounded by* $y = x^2$ *and* $y = \sqrt{x}$ *about the* $x-$ *axis.*



Solution: First we solve these two equation to find the interval of integration. Easy to see that (real) solution of $y = x^2, y = \sqrt{x}$ is $x = 0, 1$. Next we can see that $y = \sqrt{x}$ is above $y = x^2$ in this interval. Hence by above formula, the required Volume is

$$V = \pi \int_0^1 (x - x^4)dx = \frac{3\pi}{10}.$$

Volume by cylindrical shells:

A cylindrical shell is the region between two concentric cylinders of same height h. It is something like top portion of "Well" above earth surface. Let r_1 be the radius of outer cylinder and r_2 be that of inner cylinder. Then the volume of this shell is

$$V = \pi(r_1^2 - r_2^2)h = 2\pi r_a th,$$

where r_a is the average radius $(r_1 + r_2)/2$ and t is thickness of shell.
Consider the solid generated by revolving $y = f(x), a \leq x \leq b$ around the y-axis. We divide the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$. The volume V of the solid may be approximated by the sum of the volumes V_i of the shells between $[x_{i-1}, x_i]$. Each shell is approximately cylindrical. Its height is $f(x_i^*)$, where $x_i^* = (x_{i-1} + x_i)/2$, the mid point. Its thickness is $(x_i - x_{i-1})/2$. Its average radius is x_i^* . Hence its volume is

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)(x_i - x_{i-1}) \to 2\pi \int_a^b x f(x) dx \quad \text{as } n \to \infty.$$

If the region is revolved about the x axis, then

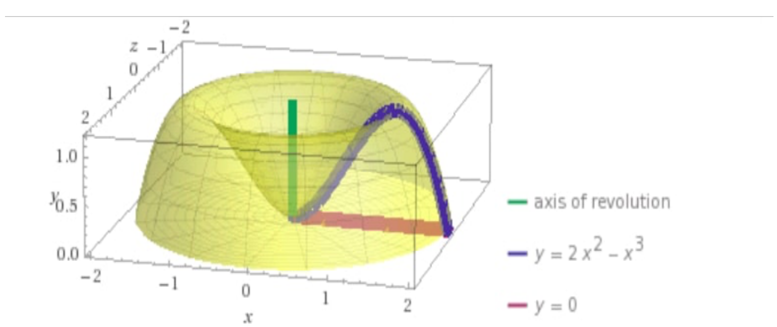
$$V = \int_a^b 2\pi y f(y) dy.$$

Suppose the solid is obtained by revolving(about y axis) the area between two curves $y = f(x)$ and $y = g(x)$ with $f(x) \geq g(x)$. Then the shell height will be $f(x_i^*) - g(x_i)^*)$. Hence the volume will be given by

$$V = 2\pi \int_a^b x(f(x) - g(x))dx.$$

Example: Find the volume obtained by revolving the area bounded by $y = 2x^2 - x^3$ and $y = 0$ about y axis.

Solution: The points of intersection of $y = 0$ and $y = 2x - x^3$ are $x = 0, 2$. Height of the shell



is $f(x) = 2x^2 - x^3$. So the volume is

$$V = \int_0^2 2\pi x f(x) = 2\pi \int_0^2 (2x^3 - x^4) = \frac{16\pi}{5}.$$

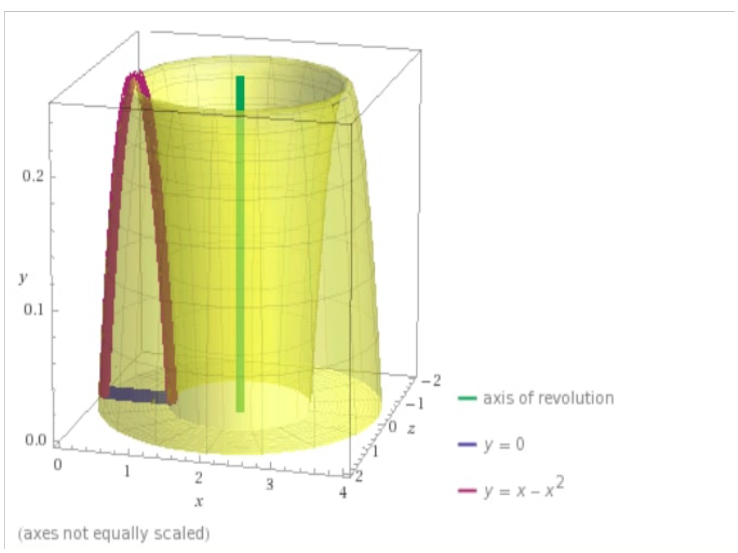
If we revolve the area about **Arbitrary line** parallel to axis , say $x = c$. Then the radius of the shell be $x - c$ (or $c - x$ whichever is positive) instead of x. So the volume in this case is

$$V = \int_a^b 2\pi(x - c)(f(x) - g(x))dx.$$

Similarly, if the region is revolved about $x = d$, then

$$V = \int_a^b 2\pi(y - d)(f(y) - g(y))dy.$$

Problem 3.0.1. *Find the volume of the solid obtained by rotating the region bounded by $y = 0$ and $y = x - x^2$ about $x = 2$.*



Solution: The points of intersection are 0 and 1. So the radius is $2 - x$ and height is $x - x^2$. Hence the volume is

$$V = \int_0^1 2\pi(2 - x)(x - x^2)dx = 2\pi \int_0^1 (x^3 - 3x^2 + 2x)dx = \frac{\pi}{2}.$$

Surface area of solids of revolution

Consider an object obtained by revolving a curve $y = f(x), a \le x \le b$ about x-axis. We assume that f is differentiable and f' is integrable. We find the surface area of this by approximating the surface by cylinders having the radius r_1 on one end and r_2 at the other end with lateral height l. The surface area of such cylinder is $2\pi\frac{r_1+r_2}{2}l$. Now we divided the interval into subintervals $[x_{i-1}, x_i]$. Let L be the line segment connecting $f(x_{i-1})$ and $f(x_i)$. Consider the small cylinders with radii $r_1 = f(x_i)$ and $r_2 = f(x_{i-1})$. Then the surface area of this cylinder is $S_i = 2\pi\frac{f(x_i)+f(x_{i-1})}{2}|L|$, where $|L|$ is the length of line segment touching $f(x_{i-1})$ and $f(x_i)$. We note as in arc length, $|L|$ is $\sqrt{1+\left(\frac{f(x_{i-1})-f(x_i)}{x_i-x_{i-1}}\right)^2}(x_i - x_{i-1})$. Hence applying mean value theorem,

$$\begin{aligned}
S &\approx \sum_{i=1}^{n} 2\pi\frac{f(x_i)+f(x_{i-1})}{2}|L| \\
&= 2\pi\sum_{i=1}^{n} \frac{f(x_i)+f(x_{i-1})}{2}\sqrt{1+\left(\frac{f(x_{i-1})-f(x_i)}{x_i-x_{i-1}}\right)^2}(x_i - x_{i-1}) \\
&= 2\pi\sum_{i=1}^{n} \frac{f(x_i)+f(x_{i-1})}{2}\sqrt{1+[f'(x_i^*)]^2}(x_i - x_{i-1}),\ x_i^* \in [x_{i-1}, x_i] \\
&\to 2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}dx.
\end{aligned}$$

Problem 3.0.2. *Find the surface area of the solid obtained by revolving the curve $y = \sqrt{4-x^2}, -1 \le x \le 1$ about x-axis.*

Solution: This is the portion of the circle $x^2 + y^2 = 4$ between $[-1, 1]$.

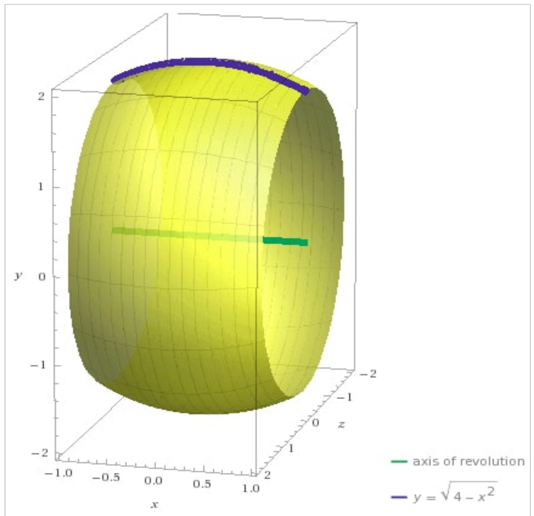
$$\begin{aligned}
S &= 2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}dx \\
&= 2\pi\int_{-1}^{1} \sqrt{4-x^2}\sqrt{1+\frac{x^2}{4-x^2}}dx \\
&= 2\pi\int_{-1}^{1} dx = 4\pi
\end{aligned}$$

If the curve is described as $x = g(y)$, then we have:

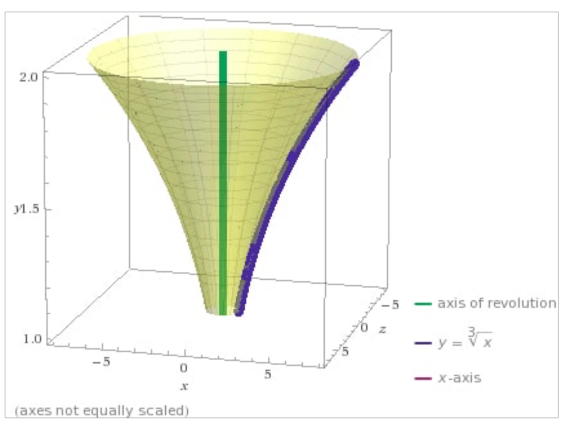
$$S = 2\pi\int_a^b y\sqrt{1+g'(y)^2}dy$$

Also, if the rotation is about the y-axis, the formula becomes,

$$S = 2\pi\int_a^b y\sqrt{1+g'(y)^2}dy.$$



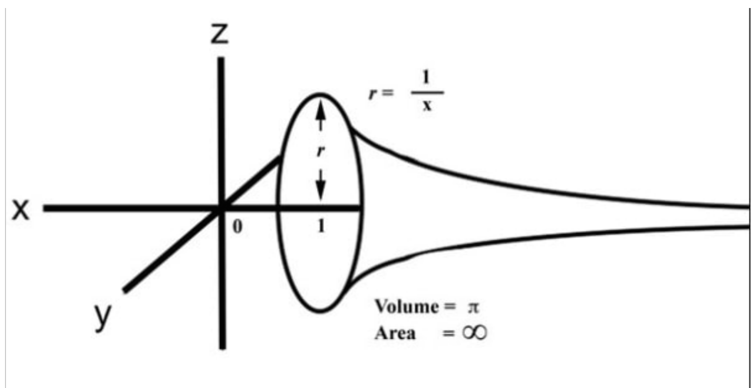
Problem 3.0.3. *Find the surface area of the solid obtained by rotating* $y = \sqrt[3]{x}, 1 \le y \le 2$ *about* y*-axis.*



Solution: Given curve is $x = y^3, 1 \leq y \leq 2$. By the given formula,

$$S = 2\pi \int_{y=1}^{2} y^3 \sqrt{1 + (\frac{dx}{dy})^2} dy = 2\pi \int_{1}^{2} y^3 \sqrt{1 + 9y^4} dy$$
$$= \frac{2\pi}{36} \int_{1}^{2} 36y^3 \sqrt{1 + 9y^4} dy = \frac{\pi}{27}(1 + 9y^4)^{3/2}\Big|_{y=1}^{2} = \frac{\pi}{27}(145^{3/2} - 10^{3/2})$$

Problem 3.0.4. *Find the surface area and volume of the solid generated by infinite curve* $y = \frac{1}{x}, x \geq 1$. *Interpret the result.*



Solution: The surface area and volume are given by

$$S = 2\pi \int_{1}^{\infty} \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx, \quad V = \pi \int_{1}^{\infty} \frac{1}{x^2} dx$$

It is easy to see that the integral in S diverges. Indeed,

$$\int_{1}^{b} \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx > \int_{1}^{b} \frac{1}{x} dx.$$

However, the integral for V converges. This is sometimes described as a can that does not hold enough paint to cover its own interior. Of course, a finite amount of paint cannot cover infinite surface. But if we fill the can with finite amount of paint we will have covered an infinite surface. This is known as **Painter's paradox.**